If the above proof is not considered "elementary," a more detailed proof, avoiding the use of the terms "harmonic range" and "polar" is as follows:

II. Extend XY to meet ARS at Z.

Then, since XZ, SP, RQ are concurrent at Y, by Ceva's Theorem,

$$\frac{SZ}{ZR}\cdot\frac{RP}{PX}\cdot\frac{XQ}{QS}=1.$$

Again, since APQ is a transversal cutting the sides of the triangle XSR, by Menelaus' Theorem,

$$\frac{SA}{AR}\cdot\frac{RP}{PX}\cdot\frac{XQ}{QS}=-1.$$

Comparing these results, we see that

$$\frac{SZ}{ZR}=-\frac{SA}{AR}.$$

Again, let TT' cut SR at Z'. From A draw AD perpendicular to TT'. On AZ' as diameter draw a circle. This circle passes through D. Bisect AZ' at O. Let one of the points of intersection of the two circles be F.

Now, $\triangle$s ACT, TCD are similar. Hence $CT^2 = CD\cdot CA$, *i. e.*, $CF^2 = CD\cdot CA$. Therefore, CF is a tangent to circle O. Hence, $\angle\ CFO$ is a right angle. Hence, OF is a tangent to circle C. Hence, $OF^2 = OR\cdot OS$, *i. e.*, $OZ'^2 = OR\cdot OS$, *i. e.*,

$$\frac{OS}{OZ'}=\frac{OZ'}{OR},$$

i. e.,

$$\frac{OS-OZ'}{OS+OZ'}=\frac{OZ'-OR}{OZ'+OR},$$

i. e.,

$$\frac{Z'S}{AS}=\frac{RZ'}{AR}.$$

Hence,

$$\frac{SZ'}{Z'R}=-\frac{SA}{AR}.$$

Hence, the points Z and Z' coincide. Therefore, the lines XY and TT' intersect ARS at the same point. Similarly it can be shown that XY and TT' intersect APQ at the same point. Hence XY and TT' coincide.

Also solved by N. P. PANDYA.

CALCULUS.

378. Proposed by ELBERT H. CLARKE, Purdue University.

The area of the curved surface generated by the revolution about OX of the portion of the curve $y = x^n$ which extends from the origin to the point (1, 1) is given by the formula

$$A = 2\pi\int_0^1 x^n\sqrt{1+n^2x^{2n-2}}\cdot dx.$$

Our geometric intuition would tell us that the limit of this area as n becomes infinite is π. Give a strict analytic proof that

$$\lim_{n\doteq\infty}\int_0^1 x^n\sqrt{1+n^2x^{2n-2}}\cdot dx=\tfrac{1}{2}.$$

SOLUTION BY ELIJAH SWIFT, University of Vermont.

We give the proof by writing the above integral as the sum of two and showing that the limit of one is zero, of the other, $\frac{1}{2}$. Let k be any value between 0 and 1, *e. g.*, .9. Then

$$\int_0^1 x^n\sqrt{1+n^2x^{2n-2}}\cdot dx=\int_0^k x^n\sqrt{1+n^2x^{2n-2}}\cdot dx+\int_k^1 x^n\sqrt{1+n^2x^{2n-2}}\cdot dx.$$

But the integrand of the first of these two integrals is less than (or equal to) $k^n\sqrt{1+n^2k^{n-2}}$ and since the limit of this is 0, the limit of the integral is zero also. To handle the second integral, write the integrand as

$$nx^{2n-1}\sqrt{1+\frac{1}{n^2x^{2n-2}}}.$$

We may now develop in series and show that the limit of every term after the first is zero, or we may proceed as follows:

$$\int_k^1 nx^{2n-1}\sqrt{1+\frac{1}{n^2x^{2n-2}}}\,dx = \int_k^1 nx^{2n-1}\left\{1+\sqrt{1+\frac{1}{n^2x^{2n-2}}}-1\right\}dx$$

$$= \int_k^1 nx^{2n-1}\cdot dx + \int_k^1 n\cdot x^{2n-1}\cdot \frac{\dfrac{1}{n^2x^{2n-2}}}{\sqrt{1+\dfrac{1}{n^2x^{2n-2}}}+1}\cdot dx,$$

by rationalizing the numerator. The limit of the first integral is $\frac{1}{2}$ and since the integrand in the second may be made as small as we please, the limit of the second integral is zero.

Also solved by J. A. Caparo and the Proposer.

379. Proposed by C. N. SCHMALL, New York City.

Express the equation of the folium, $x^3 + y^3 = 3axy$, in parametric form and find the area of the loop.

(From E. B. Wilson's *Advanced Calculus*, p. 296, ex. 5.)

Solution by E. B. Wilson, Mass. Institute of Technology.

Let $y = mx$, then

$$x = \frac{3am}{1+m^3},\qquad y=\frac{3am^2}{1+m^3};$$

the loop being described by values of m from 0 to ∞. By the formulas for area as a curvilinear integral

$$A = -\int_{m=0}^{\infty} ydx = -\int_0^\infty 9a^2\frac{m^2(1-2m^3)dm}{(1+m^3)^3} = -\int_0^\infty 3a^2\frac{1-2u}{(1+u)^3}du,$$

where $u = m^3$. Then

$$A = -3a^2\left[\frac{2}{1+u}-\frac{3}{2}\frac{1}{(1+u)^2}\right]_0^\infty = \frac{3}{2}a^2.$$

Also solved by Elijah Swift, C. E. Horne, Wilson L. Miser, W. C. Eells, Horace Olson, J. A. Caparo, H. L. Agard, L. G. Weld, and the Proposer.

MECHANICS.

297. Proposed by C. N. SCHMALL, New York City.

A shrapnel shell strikes the ground and then explodes, dispersing its fragments in all directions with a common velocity v. If a be the area of the ground covered by the fragments, and if the dimensions of the shell be neglected, show that $a = \pi v^4/g^2$.

Solution by Horace Olson, Chicago, Illinois.

According to the laws of physics, the range of a projectile on the horizontal plane from which it is thrown is $(2v^2\sin\theta\cos\theta)/g$ or $(v^2\sin 2\theta)/g$, θ being the inclination with the horizontal of the line of projection. This range has a maximum value, v^2/g, when θ is 45°. Therefore the area of the ground covered by the fragments is $\pi v^4/g^2$, the area of a circle of radius v^2/g.

Also solved by A. M. Harding, J. L. Riley, and P. Peñalver.